Polynomial Transformations

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August 2020

Before we begin, the prerequisite to this handout is knowledge of Vieta's Formulas. If you need to learn that, we have another handout on that topic here.

§1 Function Transformations

Definition 1.1. Compared to the graph of $f(x)$, the function $f(x-c)$ is shifted right by c units where negative c implies a leftwards shift.

Definition 1.2. Compared to the graph of $f(x)$, the function $f(\frac{x}{c})$ is horizontally stretched by a factor of c where $|c| < 1$ implies a compression and $c < 0$ implies an additional reflection across the y-axis.

You have likely learned the basics of function transformations within class, so I will spare you the details. There are, however, some important takeaways from these two definitions.

The most important is the intuition. Let's say we have two functions $f(x)$ and $f(x-5)$. Now, take any point (a, b) such that $f(a) = b$. To show that $f(x-5)$ is shifted right, we know that the point $(a+5, b)$ must work because when we plug it into $f(x-5)$, we get $f(a+5-5) = f(a) = b$ which is true. Since the point $(a+5, b)$ is to the right of (a, b), the function has been shifted right. Essentially, we have to "compensate" for the -5 by increasing the x-values of the function by 5, shifting it right.

This concept of "compensating" for the transformation is very useful as it allays the confusion over why we subtract 5 within the function instead of add it in order to shift right. Now, why does adding 5 outside the function *increase* the y-values?

Problem 1.3 — Show that, compared to the graph of $f(x)$, the function $f(x) + 5$ is shifted up by 5 units.

Solution. Instead of having $f(x) + 5$ be floating in air, let's turn it into the equation $f(x) + 5 = y$. Now, rearrange the equation to get $f(x) = y - 5$. It turns out that the $+5$ was actually a minus -5, just applied to the y instead of the x. Thus, we have to compensate by increasing the y-values by 5, shifting the function up. The reason we cannot say the same for the x-values is because of the function. Shifting the x-values by a constant amount will not necessarily shift $f(x)$ by a constant amount, so it would not always compensate correctly. I leave you with one more problem on this topic. $\square$

Problem 1.4 — Show that shifting the function $f(x) = x$ right by c units is the same as shifting the function down by c units.

§2 The Basics

Most polynomial transformation problems have you transform the polynomial (obviously), then apply a technique such as Vieta's Formulas. This is best shown through example.

> **Problem 2.1** — Given that α and β are the roots of $x^2 - 17x + 21 = 0$, compute $(\alpha + 1) + (\beta + 1)$.

Solution. This is a bit of an inane example as we can directly use Vieta's Formulas, but humor me. Imagine we shift the polynomial function right by 1 unit. Then, the roots of the graph would be increased by 1, which is exactly what we want. So, let's shift the function to get $(x-1)^2 - 17(x-1) + 21 = 0$. The roots of this function are $\alpha + 1$ and $\beta + 1$ (plug them in to verify), so we can try using Vieta's Formulas now. Expanding, the polynomial becomes $x^2 - 19x + 39 = 0$. Thus, the answer is just $\boxed{19}$. □

> **Problem 2.2** — Given that α and β are the roots of $x^2 - 21x + 3 = 0$, compute $\frac{1}{\alpha} + \frac{1}{\beta}$.

Solution. Uh oh, how do we transform this one? In the previous example, we used the graphical function transformation to solve it, but our main takeaway should instead be that we compensated for the $+1$ by altering x with a -1. Let's use similar reasoning here. To compensate for $\frac{1}{x}$, we want to take the reciprocal since $\frac{1}{\frac{1}{x}} = x$, just like how $(x+1) - 1 = x$. Substituting this into the function we get $\frac{1}{x^2} - \frac{21}{x} + 3 = 0$. Next, multiply both sides of the equation by x to get $3x^2 - 21x + 1 = 0$ (this works because this neither adds nor removes any solutions!). By Vieta's Formulas, the answer is $\frac{21}{3} = \boxed{7}$. Now, let's really compose some functions. □

> **Problem 2.3** — Given that α and β are the roots of $x^2 - 21x + 3 = 0$, compute $\frac{1}{\alpha+1} + \frac{1}{\beta+1}$.

Solution. Do you see why I wanted you to humor me? Just like in the last problem, let's try to find a way to compensate for $\frac{1}{x+1}$. A good first step is to take the reciprocal which leaves us with $x + 1$. Now, we can subtract by 1 which leaves just x as desired. So, the transformation we are going to use is $\frac{1}{x} - 1$. Note that this is the inverse of $\frac{1}{x+1}$ (can you see why?). After substituting, we should get $\frac{1}{x^2} - \frac{23}{x} + 25 = 0$, which results in $25x^2 - 23x + 1 = 0$. By Vieta's Formulas, our answer is $\boxed{\frac{23}{25}}$. □

§3 More Advanced Uses

> **Problem 3.1** — Given that a, b, c are the roots of $x^3 - 21x^2 + 3x + 2 = 0$, compute $(a+1)(b+1)(c+1)$

Solution. Not too far from the basics. We could just expand it, but let's transform it with $x - 1$. Then we have $(x-1)^3 - 21(x-1)^2 + 3(x-1) + 2 = x^3 - 24x^2 + 48x - 23$. Thus, we can just use Vieta's Formulas to see that the answer is $\boxed{23}$. □

Problem 3.2 — Given that α and β are the roots of $x^2 - 21x + 3 = 0$, compute $\alpha^2 + \beta^2$.

Solution. Let's try polynomial transformation. The transformation desired is $\pm\sqrt{x}$, so we get $x \mp 21\sqrt{x} + 3 = 0$. As you can see, we are stuck since we no longer have a polynomial and have no way to make it one, so Vieta's Formulas do not apply. In this situation, we must use some more creativity. Note that the roots of $(-x)^2 - 21(-x) + 3 = 0$ are exactly negative those of $x^2 - 21x + 3$, but the sign is cancelled by the squaring within the desired expression. So now, if we multiply the two polynomials together, we get $x^4 - 435x^2 + 9 = 0$. We can apply the transformation now and get $x^2 - 435x + 9 = 0$, so the answer is just $\boxed{435}$. Notice how we added two extraneous solutions by multiplying the polynomials together, then removed them during the transformation. $\square$

Problem 3.3 — Compute the following expression:

$$\sum_{n=0}^{4} \frac{1}{e^{\frac{2ni\pi}{5}} + 1}$$

Solution. There isn't even a polynomial! How are we supposed to use *polynomial* transformation? Well, note that $e^{\frac{2ni\pi}{5}}$ are the roots to the polynomial $x^5 - 1 = 0$. That means that if we can isolate $e^{\frac{2ni\pi}{5}}$, we can use this polynomial to solve the problem. The desired transformation is $\frac{1}{x} - 1$ (see Problem 2.3), so we have:

$$\left(\frac{1}{x} - 1\right)^5 = 1$$
$$(1 - x)^5 = x^5$$
$$2x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1 = 0$$

From here, we use Vieta's Formulas to achieve $\boxed{\frac{5}{2}}$. $\square$